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# THE CONNECTIVITIES OF LOCALLY FINITE PRIMITIVE GRAPHS

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Let $\Gamma$ be an infinite, locally finite graph whose automorphism group is primitive on its vertex set. It is shown that the connectivity of $\Gamma$ cannot equal 2, but all other values 0, 1, 3, 4, ... are possible.

## 1. Introduction and affirmative results

For a simple finite or infinite graph $\Gamma$, a *lobe* of $\Gamma$ is a connected subgraph of $\Gamma$ whose edge set is an equivalence class of edges with respect to the equivalence relation $\sim$, where $e\sim e'$ whenever edges $e$ and $e'$ lie on an elementary circuit in $\Gamma$. If $\Lambda$ is a lobe of $\Gamma$, then $\Lambda$ is either a maximal biconnected subgraph of $\Gamma$ or $\Lambda$ is isomorphic to $K_2$ (cf. [3, Chapter 5]). (Since this paper will involve primitive permutation groups, we prefer to reserve the term *block*, occasionally used to mean lobe, for its more traditional role.) A graph will be called *primitive* if its automorphism group acts primitively on its vertex set.

The authors' earlier characterization of primitive graphs $\Gamma$ with connectivity $\varkappa(\Gamma)=1$ will be an important tool in this paper. For completeness we state a reformulation of that result:

**Proposition 1.1** [5, Theorem 4.2]. *Let $\Gamma$ have connectivity $\varkappa(\Gamma)=1$ and be distinct from $K_2$. A necessary and sufficient condition for $\Gamma$ to be primitive is that each vertex lie on the same number of lobes of $\Gamma$ and that these lobes be themselves primitive graphs, distinct from $K_2$, and pairwise isomorphic.*

The primitive graphs described in Proposition 1.1 easily become building blocks for primitive graphs with connectivity $\varkappa\geqq 3$ as we now illustrate. Let $\Gamma=(V, E_1)$ be a locally finite primitive graph with $\varkappa(\Gamma)=1$. Letting $d$ denote the distance function in $\Gamma$, we define for $j\geqq 2$ the sets of unordered pairs of vertices

$$E_j = \{\{x, y\}\colon\ x, y\in V;\ d(x, y) = j\}.$$

If $2\leqq j_1<j_2<\ldots<j_k$ for some positive integer $k$, let $\Gamma'$ be the graph $(V, E_1\cup E_{j_1}\cup\ldots\cup E_{j_k})$. Since every automorphism of $\Gamma$ is also an automorphism of $\Gamma'$, clearly $\Gamma'$ is also primitive, and it is clear that $\varkappa(\Gamma')\geqq 3$. As a special case, suppose each vertex of $\Gamma$ is incident with exactly two lobes, the lobes being copies of the complete graph $K_n$ where $n\geqq 3$. Then one easily verifies that if $\Gamma'=(V, E_1\cup E_2)$, then $\varkappa(\Gamma')=n$.

Here is a trivial observation: an edgeless infinite graph is a 0-connected primitive graph.

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We have proved:

**Proposition 1.2.** *For every* $n=0, 1, 3, 4, 5, \ldots$ *there exists an infinite, locally finite, primitive graph whose connectivity equals n.*

Absent from this list of values is the integer 2. The remainder of this paper will be devoted to proving the following:

**Theorem 1.3.** *There exists no infinite, locally finite, primitive graph whose connectivity equals 2.*

Section 2 will provide terminology and notation needed for this proof. Section 3 will consist mainly of the proof itself. The brief final section includes a remark about infinite connectivity.

## 2. Terminology

A graph will be indicated by a pair $\Gamma=(V, E)$, where $V$ is understood to be the vertex set of $\Gamma$ and $E$ denotes its edge set. Its automorphism group, understood to act on its vertex set, will be denoted by $G(\Gamma)$. If $X$ is a subgraph of $\Gamma$, then $V(X)$ denotes the vertex set of $X$. Nearly all graphs $\Gamma$ considered here will be vertex-transitive, which implies that all vertices have equal valence; we denote that valence by $\varrho(\Gamma)$. The distance function is denoted by $d$; the graph or subgraph in which it is being applied will be evident from the context.

Let $\Gamma=(V, E)$ and let $S\subset V$. Then $\Gamma\setminus S$ indicates the subgraph of $\Gamma$ induced by the vertex subset $V\setminus S$. Thus, if $x, y\in V$, we understand $\Gamma\setminus\{x, y\}$ to be the subgraph remaining when $x$ and $y$ and all edges incident with either of them are deleted from $\Gamma$. (In particular, when $\{x, y\}\in E$, then $\Gamma\setminus\{x, y\}$ is obtained by removing more than just one edge!) If $C$ is a component of $\Gamma\setminus S$, then $\bar{C}$ denotes the subgraph of $\Gamma$ induced by all vertices in neither $S$ nor $C$. Following [4] we define for infinite graphs $\Gamma$

$$\varkappa_f(\Gamma) = \inf_{S\subset V} \{|S|: \Gamma\setminus S \text{ has at least one finite component}\}$$

and

$$\varkappa_\infty(\Gamma) = \inf_{S\subset V} \{|S|: \Gamma\setminus S \text{ has at least two infinite components}\}.$$

Thus the usual connectivity of an infinite graph $\Gamma$ is

$$\varkappa(\Gamma) = \min \{\varkappa_f(\Gamma), \varkappa_\infty(\Gamma)\}.$$

When $\Gamma$ is finite, set $\varkappa_f(\Gamma)=\varkappa(\Gamma)$.

By a *slice* of $\Gamma$ we will mean any finite subset of $V$ whose deletion from $\Gamma$ leaves at least two infinite components. Thus $\varkappa_\infty(\Gamma)$ when finite is the cardinality of a smallest slice. We will require:

**Proposition 2.1** [6, Lemma 2.4]. *Let* $\Gamma$ *be infinite and locally finite, and let* $\tau\in G(\Gamma)$. *There exists a subset* $U\subset V$ *such that* $\tau[U]\subset U$ *(N. B. proper containment!) if and only if* $\tau[T]\neq T$ *for every nonempty finite subset* $T\subset V$.

If $\Gamma$ is vertex-transitive, its components are blocks of imprimitivity. If $\Gamma$ is bipartite and connected, its bipartition yields a system of imprimitivity. Thus if $\Gamma$ is primitive and $|V|>2$, then $\Gamma$ is connected and not bipartite.

## 3. Primitive graphs of connectivity 2

**Lemma 3.1.** *If $\Gamma$ is a primitive gaph with $\varkappa_f(\Gamma)=2$, then $\Gamma$ is a circuit of prime length.*

**Proof.** If $\Gamma$ is primitive it is vertex-transitive. For any separating set $S$ of cardinality $\varkappa_f(\Gamma)$, the finite components of $\Gamma\setminus S$ are called "$\varkappa_f$-fragments". A $\varkappa_f$-fragment with the least number of vertices is called an *atom*. Since the atoms of $\Gamma$ form a system of imprimitivity (see [7, Theorem 2] or [2]), they must be singleton sets. Hence $\varrho(\Gamma)=\varkappa_f(\Gamma)=2$ (cf. [7, Lemma 2.1]). Since $\Gamma$ is connected, it is a circuit, necessarily of prime length. ∎

The next lemma is somewhat more general than is needed for the sequel.

**Lemma 3.2.** *Let $\Gamma=(V, E)$ be a graph with $\varkappa_\infty(\Gamma)=2$. For every vertex $u$ of finite valence, there exists a vertex $v$ such that $\{u, v\}\in E$ and $\{u, v\}$ is not a slice of $\Gamma$.*

**Proof.** Suppose that for some vertex $u$ of finite valence, $\{u, v\}$ is a slice whenever $\{u, v\}\in E$. Among all the pairs $(v, C)$ such that $\{u, v\}\in E$ and $C$ is an infinite component of $\Gamma\setminus\{u, v\}$, select a pair $(v', C')$ such that $C'$ contains the fewest neighbors of $u$. Since $\{v'\}$ is not a slice of $\Gamma$, $C'$ must contain a neighbor $v''$ of $u$. Let $C''$ be an infinite component of $\Gamma\setminus\{u, v''\}$ which does not contain $v'$. Since $C''\cup\{v''\}$ induces a connected subgraph disjoint from $\{u, v'\}$, we have $C''\subset C'$. Thus $C''$ contains even fewer neighbors of $u$ than $C'$ does, contrary to the choice of $(v', C')$. ∎

Theorem 1.3 will be an immediate consequence of Lemma 3.1 and the following result.

**Theorem 3.3.** *There exists no infinite, locally finite, primitive graph $\Gamma$ with $\varkappa_\infty(\Gamma)=2$.*

This theorem will follow from the ensuing sequence of lemmas. These lemmas are interspersed with notational conventions to be continued from one lemma to the next. We begin by assuming $\Gamma=(V, E)$ to be an infinite, locally finite, primitive graph with $\varkappa_\infty(\Gamma)=2$. Among all the minimum slices of $\Gamma$, let $\{x_1, x_2\}$ be one such that $d(x_1, x_2)$ is minimal. By Lemma 3.1, $\varkappa_f(\Gamma)>2$, and so all components of $\Gamma\setminus\{x_1, x_2\}$ are infinite. Let

$$F=\{\{\varphi(x_1), \varphi(x_2)\}:\ \varphi\in G(\Gamma)\}.$$

Clearly $F$ is fixed setwise by $G(\Gamma)$. Let $\Gamma_F=(V, F)$ and $\Gamma'=(V, E\cup F)$. (In the special case that $d(x_1, x_2)=1$ we have $F\subseteq E$ and $\Gamma=\Gamma'$.) Since $G(\Gamma)\leqq G(\Gamma_F)$ and $G(\Gamma)\leqq G(\Gamma')$, both $\Gamma_F$ and $\Gamma'$ are primitive graphs.

**Lemma 3.4.** *If $\{x, y\}\in F$, then the components of $\Gamma\setminus\{x, y\}$ and the components of $\Gamma'\setminus\{x, y\}$ have the same vertex sets.*

**Proof.** If $d(x_1, x_2)=1$ there is nothing to prove. We assume that $d(x_1, x_2)\geqq 2$ and that for some component $C_1$ of $\Gamma\setminus\{x_1, x_2\}$ there exists in $\Gamma'$ an edge $\{y_1, y_2\}\in F$ with $y_1\in V(C_1)$ and $y_2\in V(\bar{C}_1)$. We assume without loss of generality that for some $\varphi\in G(\Gamma)$ one has $\varphi(x_i)=y_i$ $(i=1, 2)$. Certainly $\{y_1, y_2\}$ is another slice of $\Gamma$. Now $\Gamma\setminus\{y_1, y_2\}$ has exactly two components, for otherwise, there would exist a $y_1 y_2$-path in $\Gamma$ which would avoid $x_1$ and $x_2$. We may therefore write $C_2$ in place of $\bar{C}_1$ and let $D_i=\varphi[C_i]$, $(i=1, 2)$. Thus $D_1$ and $D_2$ are the two components of $\Gamma\setminus\{y_1, y_2\}$; $x_1$

lies in one of them and $x_2$ lies in the other. Hence

$$d(x_1, x_2) = \min \{d(x_1, y_j)+d(y_j, x_2)\colon j = 1, 2\}.$$

For definiteness, suppose the above minimum is obtained when $j=1$.

We now obtain a contradiction to the minimality of $d(x_1, x_2)$ by showing that $\{x_1, y_1\}$ or $\{x_2, y_1\}$ is a slice of $\Gamma$. Since $C_1 \cap D_1$ or $C_1 \cap D_2$ is infinite, we assume without loss of generality that $C_1 \cap D_1$ is infinite. Let $w_1 \in C_1 \cap D_1$ and $w_2 \in C_2 \cup D_2$.

**Case A:** $x_1 \in D_1$ and $x_2 \in D_2$. Suppose that $\Gamma$ contains a $w_1 w_2$-path $\Pi$ that avoids $x_1$ and $y_1$.

**Subcase A1:** $w_2 \in C_2$. Since $\Pi$ is a $C_1 C_2$-path it must pass through $x_2$. Since $\Pi[w_1, x_2]$ is a $D_1 D_2$-path, it must pass through $y_2$. But $\Pi[w_1, y_2]$ is a $C_1 C_2$-path avoiding $x_1$ and $x_2$.

**Subcase A2:** $w_2 \in D_2$. Since $\Pi$ is a $D_1 D_2$-path, it must pass through $y_2$, and we proceed similarly.

**Case B:** $x_1 \in D_2$ and $x_2 \in D_1$. Let $\Pi$ be a $w_1 w_2$-path that avoids $x_2$ and $y_1$, and proceed analogously. ∎

A consequence of the above Lemma is that $\varkappa_\infty(\Gamma')=2$. Also $\Gamma'$ is locally finite since for every $\Gamma_F$-neighbor $y_1$ of $x_1$ one has $d(x_1, y_1)=d(x_1, x_2)$ in $\Gamma$.

**Lemma 3.5.** $\varkappa_\infty(\Gamma_F)=1$.

**Proof.** Since $\Gamma_F$ is primitive, it is connected. Since $\Gamma_F$ is a spanning subgraph of $\Gamma'$ every slice of $\Gamma'$ contains a slice of $\Gamma_F$. By Lemma 3.4, some pair of adjacent vertices of $\Gamma_F$ is a slice of $\Gamma'$. Were it also a slice of $\Gamma_F$, then every pair of adjacent vertices in $\Gamma_F$ would be a slice of $\Gamma_F$, contrary to Lemma 3.2. ∎

By Proposition 1.1, we know the general structure of $\Gamma_F$. Let $\Lambda_0$ be the lobe of $\Gamma_F$ which contains $\{x_1, x_2\}$. We remark that $G(\Gamma)$ acts edge-transitively on $\Gamma_F$ and that if some $\varphi \in G(\Gamma)$ maps an edge of $\Lambda_0$ onto an edge of $\Lambda_0$, then $\varphi[\Lambda_0]=\Lambda_0$. Hence some subgroup of $G(\Gamma)$ acts edge-transitively on $\Lambda_0$.

**Lemma 3.6.** *$V(\Lambda_0) \setminus \{x_1, x_2\}$ is contained in one component of $\Gamma \setminus \{x_1, x_2\}$.*

**Proof.** To negate this assertion is to assert that $\{x_1, x_2\}$ separates $\Lambda_0$, and so $\varkappa(\Lambda_0)=2$ but $\Lambda_0$ is not a circuit. By Proposition 1.1, $\Lambda_0$ is primitive. Hence $\varkappa_f(\Lambda_0)>2$ by Lemma 3.1. If $\varkappa_\infty(\Lambda_0)=2$, the edge-transitivity of $\Lambda_0$ yields a contradiction to Lemma 3.2. ∎

Henceforth let $C_0$ denote the component of $\Gamma \setminus \{x_1, x_2\}$ that contains $\Lambda_0 \setminus \{x_1, x_2\}$.

**Lemma 3.7.** *The restriction of $G(\Gamma)$ to any lobe of $\Gamma_F$ acts vertex-transitively.*

**Proof.** Let $G_0$ be the restriction of $G(\Gamma)$ to $\Lambda_0$. We have remarked (before Lemma 3.6) that $G_0$ acts edge-transitively. If $G_0$ did not act vertex-transitively, then by a standard theorem of graph theory, $\Lambda_0$ would be bipartite, contrary to the fact that $\Lambda_0$ is primitive by Proposition 1.1. ∎

For $x \in V$, we define

$$F^+(x) = \{\varphi(x_2)\colon \varphi \in G(\Gamma) \text{ and } \varphi(x_1) = x\}.$$

and

$$F^-(x) = \{\varphi(x_1): \varphi \in G(\Gamma) \quad \text{and} \quad \varphi(x_2) = x\}.$$

In particular, $x_2 \in F^+(x_1)$ and $x_1 \in F^-(x_2)$.

**Lemma 3.8.** *Among the components of $\Gamma \setminus \{x_1, x_2\}$ at most one contains a vertex in $F^+(x_1)$ and at most one contains a vertex in $F^-(x_2)$.*

**Proof.** We include a proof only for $F^+(x_1)$, the proof for $F^-(x_2)$ being analogous. Let $r_1, r_2 \in F^+(x_1)$ and suppose that $R_1$ and $R_2$ are distinct components of $\Gamma \setminus \{x_1, x_2\}$ containing $r_1$ and $r_2$, respectively. By definition, there exist $\varphi_i \in G(\Gamma)$ such that $\varphi_i(x_1) = x_1$ and $\varphi_i(x_2) = r_i$.

Since $V(\bar{R}_1) \cup \{x_2\}$ induces a connected subgraph $\Theta$ of $\Gamma \setminus \{x_1, r_1\}$, we must have $\Theta \subseteqq \varphi_1[R]$ for some component $R$ of $\Gamma \setminus \{x_1, x_2\}$. In fact, $R = R_1$, for otherwise we would have $\Theta \subseteqq \varphi_1[\bar{R}_1]$, which implies that $\varphi_1^{-1}$ maps $\bar{R}_1$ properly into itself, contrary to Proposition 2.1. By complementation:

$$\varphi_1[V(\bar{R}_1) \cup \{x_2\}] \subseteqq V(R_1). \tag{1}$$

Symmetrically one has

$$\varphi_2[V(\bar{R}_2) \cup \{x_2\}] \subseteqq V(R_2). \tag{2}$$

Clearly

$$R_1 \subseteqq \bar{R}_2 \quad \text{and} \quad R_2 \subseteqq \bar{R}_1. \tag{3}$$

Define $\varphi_3 = \varphi_1 \varphi_2$. By applying successively (3), (2), (3) and (1), we obtain

$$\begin{aligned} \varphi_3[V(R_1)] &\subset \varphi_3[V(\bar{R}_2) \cup \{x_2\}] \\ &\subseteqq \varphi_1[V(R_2)] \\ &\subset \varphi_1[V(\bar{R}_1) \cup \{x_2\}] \\ &\subseteqq V(R_1). \end{aligned}$$

However, $\varphi_3(x_1) = x_1$ in contradiction to Proposition 2.1. ∎

**Lemma 3.9.** *Each lobe $\Lambda$ of $\Gamma_F$ is a circuit of prime length and $G(\Gamma)$ acts cyclically on $V(\Lambda)$.*

**Proof.** By Lemma 3.7, for each $x \in V$ and each lobe $\Lambda$ of $\Gamma_F$ containing $x$, the sets $V(\Lambda) \cap F^+(x)$ and $V(\Lambda) \cap F^-(x)$ are nonempty. We first show that either $V(\Lambda_0) \cap \cap F^+(x_1) = \{x_2\}$ or $V(\Lambda_0) \cap F^-(x_2) = \{x_1\}$.

Suppose there exists a vertex $z \in V(\Lambda_0) \cap (F^+(x_1) \setminus \{x_2\})$. Since $z \in V(C_0)$, we have by Lemma 3.8 that $F^+(x_1) \cap V(\bar{C}_0) = \emptyset$. Thus $x_1$ belongs to no lobe of $\Gamma_F$ that meets $\bar{C}_0$, and hence, since $\Gamma_F$ is connected, $x_2$ must belong to such a lobe $\Lambda_1$. That is, $V(\bar{C}_0) \cap F^-(x_2) \cap V(\Lambda_1) \neq \emptyset$. By Lemma 3.8, $V(C_0) \cap F^-(x_2) = \emptyset$. Hence $V(\Lambda_0) \cap F^-(x_2) = \{x_1\}$.

Without loss of generality we may assume that $V(\Lambda_0) \cap F^-(x_2) = \{x_1\}$. By Lemma 3.7 it follows that $|V(\Lambda_0) \cap F^-(x)| = 1$ for all $x \in V(\Lambda_0)$. Let $\Delta$ be a circuit in $\Lambda_0$ through the edge $\{x_1, x_2\}$. We may write $V(\Delta) = \{x_1, x_2, \ldots, x_p\}$, where $p \geqq 3$, $V(\Lambda_0) \cap F^-(x_i) = \{x_{i-1}\}$, and subscripts are read modulo $p$. If $\Lambda_0 = \Delta$ there is nothing more to prove. Otherwise we must reconsider the vertex $z$ whose existence was hypothesized above. Since $\Lambda_0$ is biconnected there exists a $z\Delta$-path in $\Lambda_0$ with vertex set $\{z = u_0, u_1, \ldots, u_{j-1}, u_j = x_k\}$ for some $x_k \neq x_1$. Since $x_1 \in F^-(u_0)$ we successively obtain $u_{i-1}\in F^-(u_i)$, $1\leqq i\leqq j$. But then $x_{k-1}, u_{j-1}\in F^-(x_k)$, giving a contradiction, since these two vertices must be distinct. ▮

We have seen that $x_1$ or $x_2$ belongs to no lobe of $\Gamma_F$ other than $\Lambda_0$ which meets $C_0$, by Lemmas 3.8 and 3.9. For definiteness, let us agree that

**3.10.** *$x_2$ belongs to no lobe of $\Gamma_F$ other than $\Lambda_0$ which has a vertex in $C_0$.*

There exists an edge $\{u, v\}\in E\setminus F$ such that $u$ and $v$ belong to no common lobe of $\Gamma_F$, for otherwise one would have $\varkappa_\infty(\Gamma)=1$.

**Lemma 3.11.** *If $\{u, v\}\in E\setminus F$ and if $u$ and $v$ belong to no common lobe of $\Gamma_F$, then $F^+(u)\cap F^+(v)\neq\emptyset$.*

**Proof.** By Lemma 3.9, there exists a unique path $\Pi$ with $V(\Pi)=\{u=u_0, u_1, ..., u_k= =v\}$ in $\Gamma_F$ such that $u_{i+1}\in F^+(u_i)$, $(i=0, 1, ..., k-1)$. Denote the successive lobes of $\Gamma_k$ having edges in $\Pi$ by $\Lambda_1, ..., \Lambda_l$. Thus $l\geqq 2$.

**Case A:** Some two edges of $\Pi$ belong to the same lobe $\Lambda_i$ for some $i\leqq l-1$. Suppose both $\{u_{h-1}, u_h\}$ and $\{u_h, u_{h+1}\}$ belong to the same lobe $\Lambda_i$ and that $u_{h+1}$ is also a vertex of $\Lambda_{i+1}$. There exists $\sigma\in G(\Gamma)$ such that $\sigma(u_h)=x_1$ and $\sigma(u_{h+1})=x_2$. Note that $h\geqq 1$ and $h+1\leqq k-1$. Thus $\sigma[\Lambda_i]=\Lambda_0$ and $\sigma(u_{h-1})\in V(C_0)\cap V(\Lambda_0)$. Hence $C_0$ contains all vertices of $\sigma[\Pi[u, u_{h-1}]]$. On the other hand, by 3.10 one has $\sigma[V(\Lambda_{i+1})]\subseteqq V(\bar{C}_0)\cup\{x_2\}$ and so by Lemma 3.4, $\bar{C}_0$ contains all vertices of $\sigma[\Pi[u_{h+2}, v]]$. Thus $\sigma(\{u, v\})$ is a $C_0\bar{C}_0$-edge, which is impossible.

**Case B:** For each $i=1, ..., l-1$, the lobe $\Lambda_i$ contains a unique edge $\{u_{i-1}, u_i\}$ of $\Pi$. Let

$$h=\begin{cases}2 & \text{if } l=2\\ l-1 & \text{if } l\geqq 3.\end{cases}$$

By Lemma 3.9 there is a unique vertex $v'\in V(\Lambda_h)\cap F^-(u_{h-1})$. If $v'=v$, then $\Pi[u_{h-1}, v]\subseteqq\Lambda_h$. Hence $h=l$ and $h=2$ by definition of $h$, and $u_1\in F^+(u)\cap F^+(v)$ as required.

Finally assume $v\neq v'$. There exists $\tau\in G(\Gamma)$ such that $\tau(v')=x_1$ and $\tau(u_{h-1})=x_2$. Thus $\tau[\Lambda_h]=\Lambda_0$, and so $\tau(u_h)$ lies in $C_0$ and, moreover, $C_0$ contains all vertices of $\tau[\Pi[u_h, v]]$. On the other hand, by 3.10 one has $\tau[V(\Lambda_{h-1})]\subseteqq V(\bar{C}_0)\cup\{x_2\}$ and so $\bar{C}_0$ contains all vertices of $\tau[\Pi[u, u_{h-2}]]$, and again a contradiction is obtained. ▮

Let $\{u, v\}\in E\setminus F$ such that $u$ and $v$ belong to no common lobe of $\Gamma_F$ and let $H$ be the edge-orbit of $\{u, v\}$ under action of $G(\Gamma)$. Let $\Gamma_H=(V, H)$. Since $G(\Gamma)\leqq \leqq G(\Gamma_H)$, it follows that $\Gamma_H$ is primitive and hence connected. Thus the following lemma will provide the ultimate contradiction to complete our proof of Theorem 3.3.

**Lemma 3.12.** *$\Gamma_H$ is not connected.*

**Proof.** Suppose that $\Gamma_H$ is connected, and let $\Sigma$ be a shortest path in $\Gamma_H$ joining two distinct vertices of some lobe of $\Gamma_F$. In the light of the action of $G(\Gamma)$ on $\Gamma'$ we may assume that $\Sigma$ joins $x_1$ to another vertex $x_j$ also on $\Lambda_0$. Let $V(\Lambda_0)= =\{x_1, x_2, ..., x_p\}$, where according to Lemma 3.9, we may assume that $V(\Lambda_0)\cap \cap F^+(x_i)=\{x_{i+1}\}$, subscripts being read modulo $p$. Suppose $\{x_1, s\}$ is an edge of $\Sigma$. By Lemma 3.11 and the definition of $H$, there exists a vertex $s'\in F^+(x_1)\cap F^+(s)$. Let $\Lambda_1$ be the lobe of $\Gamma_F$ containing $\{s, s'\}$. Clearly $\Lambda_1\neq\Lambda_0$ since $s\notin V(\Lambda_0)$.

**Case A:** $s'=x_2$. By 3.10, $V(\Lambda_1)\subseteqq V(\bar{C}_0)\cup\{x_2\}$. In particular, $s\in V(\bar{C}_0)$, and we have a contradiction unless $\Sigma[s, x_j]$ passes through $x_2$, which means that in fact $x_j=x_2$. But then $\Sigma[s, x_2]$ joins two vertices of a common lobe, namely $\Lambda_1$, and contradicts the minimality of the length of $\Sigma$.

**Case B:** $s'\neq x_2$.

**Subcase B1:** $x_j\neq x_p$. One may choose $\tau\in G(\Gamma)$ so that $\tau(x_1)=x_2$ and $\tau(x_2)=x_3$. Clearly $\tau$ advances each vertex of $\Lambda_0$ around the circuit. By the same reasoning as in Case A and since $\tau(s')\in F^+(x_2)$, we have $\tau(s')\in V(\bar{C}_0)$. By Lemma 3.4, $\tau(s)\in V(\bar{C}_0)$ also, while $\tau(x_j)$ remains in $V(C_0)$. However, $\tau[\Sigma[s, x_j]]$ passes through neither $x_1$ nor $x_2$, which is impossible.

**Subcase B2:** $x_j=x_p$. Let $\{x_j, t\}$ be an edge of $\Sigma$. By Lemma 3.11, there exists a vertex $t'\in F^+(x_j)\cap F^+(t)$. It $t'=x_1$, use the automorphism $\tau$ of Subcase B1 and apply the argument of Case A to the images under $\tau$. If $t'\neq x_1$, then $\tau[\Sigma]$ is a path in $\Gamma_H$ joining $x_1$ to $x_2$, which reduces to Subcase B1. ∎

## 4. Concluding remark

Let $\Gamma$ be any primitive graph with $\varkappa(\Gamma)=1$, as characterized by Proposition 1.1. If $\Gamma\neq K_2$ and $\Gamma$ is locally finite, then indeed $\varkappa_\infty(\Gamma)=1$. Since $\Gamma$ is prime with respect to the strong product, it follows by a theorem of *W.* Dörfler and W. Imrich [1] that the strong product $\Gamma*\Gamma$ of $\Gamma$ with itself is also a primitive graph. Moreover, $\varkappa_\infty(\Gamma*\Gamma)=\infty$.

Combining this result with the examples in Section 2 and Theorem 3.3, we have:

**Theorem 4.1.** *Let $\Gamma$ be a locally finite, primitive graph with $E(\Gamma)\neq\emptyset$. Then $\varkappa_\infty(\Gamma)$ may equal* 1, 3, 4, ..., ∞, *but may not equal* 2.

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